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Some V_4-cordial graphs

M. Seenivasan[1] **and A. Lourdusamy**[2*]
[1]Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627412, India
Email: msvasan_22@yahoo.com

[2]Department of Mathematics, St. Xavier's College (Autonomous),
Palayamkottai–627002, India. Email:lourdugnanam@hotmail.com

Abstract. For any abelian group A, a graph G is said to be A-cordial if there exists a labeling $f : V(G) \to A$ such that for every $a, b \in A$ we have

(1) $|v_a - v_b| \leq 1$ and (2) $|e_a - e_b| \leq 1$, where v_a and e_a respectively denote the number of vertices and edges having particular label a. In the present work we investigate a necessary condition for an Eulerian graph to be V_4-cordial. In addition to this we show that all trees except P_4 and P_5 are V_4-cordial and the cycle C_n is V_4-cordial if and only if $n \neq 4$ or 5 or $n \not\equiv 2 \pmod 4$.

Keywords: Graph labeling, A-cordial, V_4-cordial.

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1 Introduction

Throughout this work by graph $G = (V(G), E(G))$ we mean a simple graph with p vertices and q edges. The terminology followed in this paper is according to [5]. A graph labeling is an assignment of labels to the vertices or edges, or both subject to certain conditions. For a summary on various graph labeling see the Dynamic survey of graph labeling by Gallian [4]. For any abelian group A, Hovey [1] introduced A-cordial labeling. According to him a graph is called A-cordial if there exists a labeling

$f : V(G) \to A$ such that for every $a, b \in A$ we have (1) $|v_a - v_b| \leq 1$ and (2) $|e_a - e_b| \leq 1$, where v_a and e_a respectively denote the number of vertices and edges having particular label a. If $A = Z_k$, the labeling is called k-cordial. The k-cordial graphs were studied in [1, 2, 3]. There are only two non-isomorphic abelian groups of order four, which are Z_4 and the Klein-four group V_4. In the present work we investigate a necessary condition for an Eulerian graph to be V_4-cordial. In addition to this we show that all trees except P_4 and P_5 are V_4-cordial and the cycle C_n is V_4-cordial if and only if $n \neq 4$ or 5 or $n \not\equiv 2 \pmod 4$.

2 V_4-cordial graphs

The Klein-four group V_4 is the direct sum $Z_2 \oplus Z_2$.

$\oplus$	**(0,0)**	**(0,1)**	**(1,0)**	**(1,1)**
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

For simplicity, we will denote (0,0),(0,1),(1,0) and (1,1) by 0, a, b, c respectively. That is $V_4 = \{0, a, b, c\}$, with $a+a = b+b = c+c = 0$, $a+b = c$, $b+c = a$, $c+a = b$, and $a+b+c = 0$.

Lemma 2.1. *[1] If f is an A-cordial labeling of G, so is $f+a$ for any $a \in A$.*

Theorem 2.2. *If G is an Eulerian graph with q edges, where $q \equiv 2 \pmod 4$, then G has no V_4-cordial labeling.*

Proof. Suppose there exists a V_4-cordial labeling, f, of an Eulerian graph G with q edges, where $q \equiv 2 \pmod 4$. Then $q = 4m+2$ for some integer m. Let the edges e_i have the edge labels b_i in in the labeling f. Evidently $\sum_{i=1}^{q} b_i = m(0+a+b+c)+x+y = x+y$, where $x, y \in \{0, a, b, c\}$ and $x \neq y$. Thus $\sum_{i=1}^{q} b_i \neq 0$. But $\sum_{i=1}^{q} b_i = d(v_i)f(v_i) = 0$ as $d(v)$, the degree of the vertex v in G, is even. This contradiction proves the theorem. ∎

Corollary 2.3. *The cycle C_n is not V_4-cordial, where $n \equiv 2 \pmod 4$, the generalized Peterson graph $P(n,k)$, where $n \equiv 2 \pmod 4$, and $C_m \times C_n$ where m and n are odd are not V_4-cordial.*

Theorem 2.4. *Let f be a V_4-cordial labeling of a graph G with $p \geq 4$ and uv be an edge of G such that $f(u) = 0$ and $f(u) \neq f(v)$. Then the graph G' obtained from G by replacing the edge uv by a path of length five is V_4-cordial.*

Proof. Let G' be a graph obtained from G by replacing the edge uv by a path $uw_1w_2w_3w_4v$. Suppose $f(v) = a$. Define $f_1 : V(G') \to V_4$ by

$$f_1(w) = \begin{cases} f(w), & \text{if } w \in V(G) \\ 0, & \text{if } w = w_1 \\ a, & \text{if } w = w_2 \\ b, & \text{if } w = w_3 \\ c, & \text{if } w = w_4. \end{cases}$$

Clearly f_1 is a V_4-cordial labeling of G'. In a similar way a V_4-cordial labeling of G can be extended to a V_4-cordial labeling of G when $f(v) = b$ or c. ∎

Theorem 2.5. *Let P_n denote the path on n vertices. Then P_4 and P_5 are not V_4-cordial.*

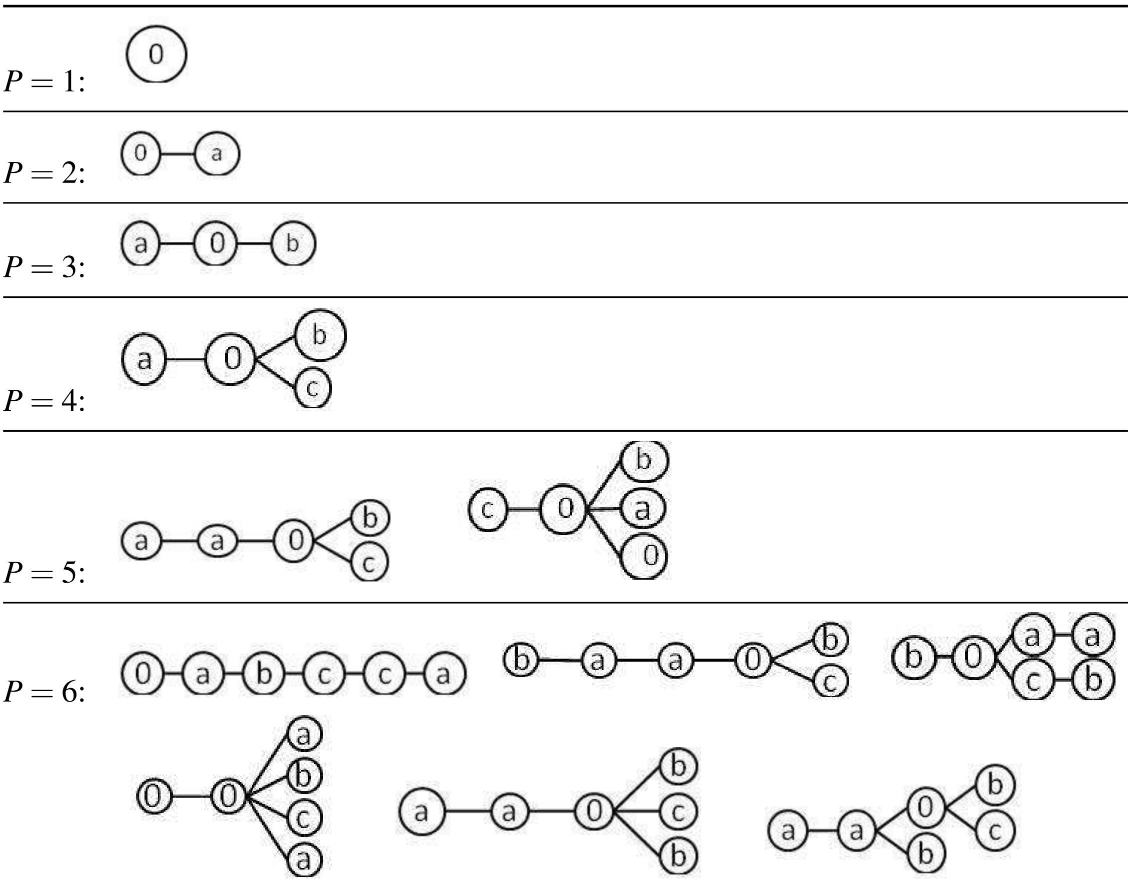
Proof. Let f be a V_4-cordial labeling of $P_4 = v_1v_2v_3v_4$. We note that the vertices of P_4 receive distinct labels under f. Without loss of generality we assume $f(v) = 0$. Then the induced edge labels of v_1v_2 and v_3v_4 are identical. This is a contradiction. Suppose f be a V_4-cordial labeling of P_5. It is clear that the induced edge labels are distinct. Let zero be the induced edge label of the edge uv. Then a V_4-cordial labeling of P_4 can be obtained by removing the edge uv from P_5 and identifying the vertex v with u. This is a contradiction. ∎

Lemma 2.6. *If all trees on $4m$ vertices are V_4-cordial then all trees on $4m+1, 4m+2, 4m+3$ vertices are also V_4-cordial.*

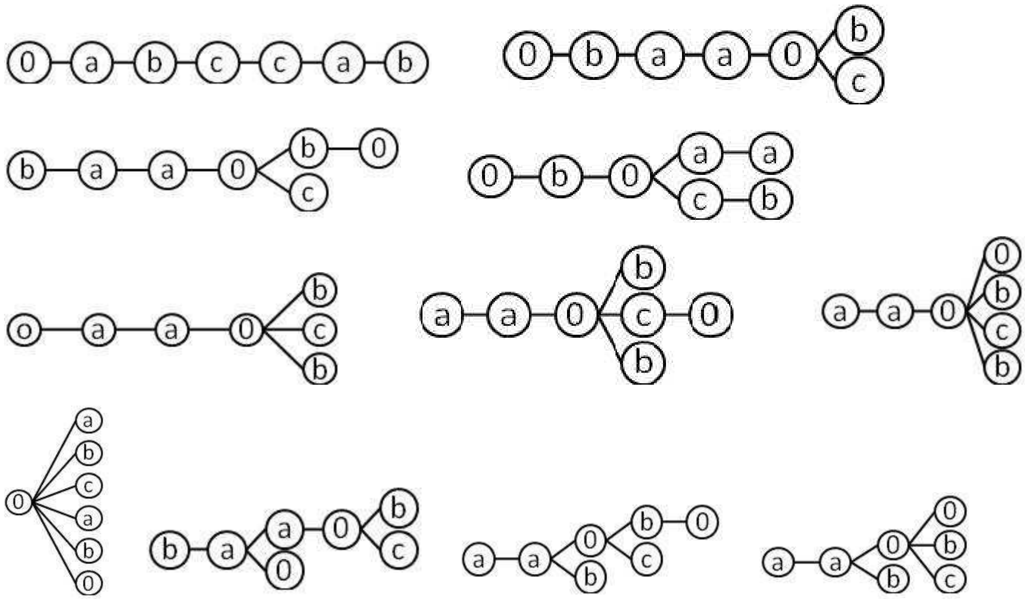
Proof. If we attach a leaf to a tree with $4m + j$ vertices we have choices for the vertex labels that will preserve vertex V_4-cordiality of the tree. In order to preserve edge V_4-cordiality we must avoid $j-1$ edge labels if $j > 0$. We can do this as long as $4 - j > j - 1$. If $j = 0$ we have only one choice for the edge label but there is no restriction on vertex labels. ∎

Theorem 2.7. *All trees except P_4 and P_5 are V_4-cordial.*

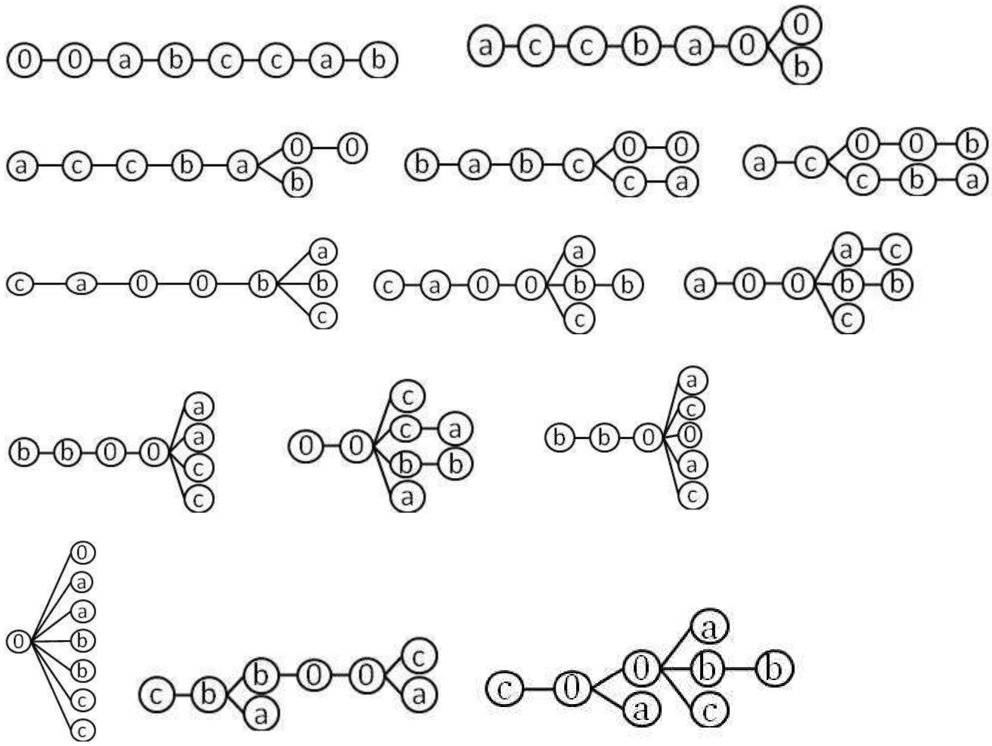
Proof. First we shall show that all trees on $p \leq 8$ vertices except P_4 and P_5 are V_4-cordial. This is verified by the labellings given in Fig. 1.



$P = 7$:



$P = 8$:



$P = 8$:

(contd.)

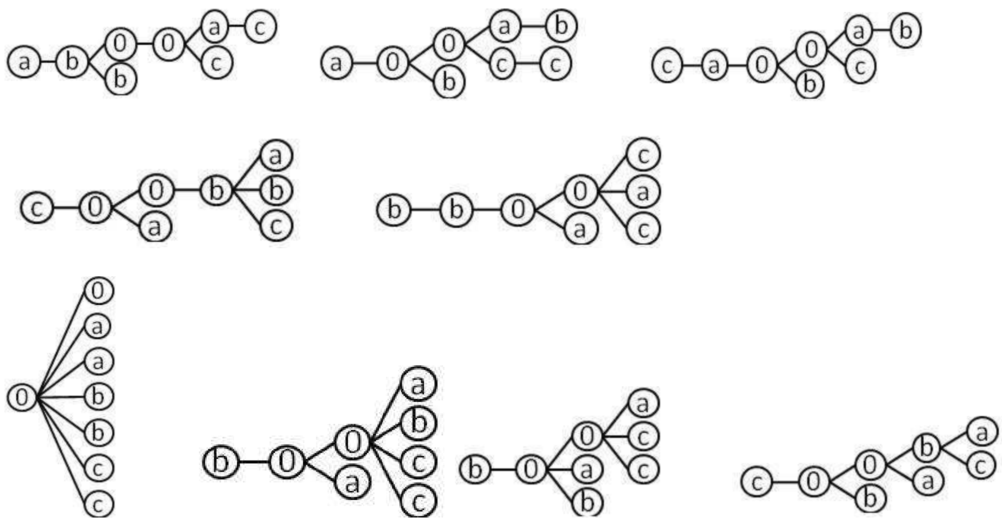


Figure 1:

Now by the Lemma 2.6, we only need to show that trees with $4m$ vertices are V_4-cordial implies trees with $4m+4$ vertices are V_4-cordial when $m \geq 2$. Let T be a tree with $4m+4$ vertices and $m \geq 2$.

Case i: T has four leaves.
Let l_0, l_1 , l_2 , l_3 be four leaves connected to v_0, v_1, v_2, v_3 respectively. Delete them and label the resulting tree V_4-cordially. Let the labels on the v_i be denoted a_i. Then we can assume, by permuting the V_i and by Lemma 2.1, that (a_o, a_1, a_2, a_3) is one of (0, 0, 0, 0), (0, 0, 0, a), (0, 0, 0, b), (0, 0, 0, c), (0, 0, a, a), (0, 0, b, b), (0, 0, c, c), (0, 0, a, b), (0, 0, a, c), (0, 0, b, c), (0, a, a, b), (0, a, a, c), (0, a, b, c). Suppose that edge-label j appears $m-1$ times while the other two edge-labels appear m times. We must find a way of labeling l_0, l_1, l_2, l_3 with distinct elements so that j appears as an edge-label and no other edge-label appears twice, though j itself might. We do this case by case. Each case is presented as an array with the top row being the a_i, the middle row the labels on the l_i and the bottom row the induced edge-labels.

$$\begin{array}{llll} 0 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & a & b & c \end{array} \qquad \begin{array}{llll} 0 & 0 & 0 & a \\ j & j+b & j+c & j+a \\ j & j+b & j+c & j \end{array}$$

$$\begin{array}{llll} 0 & 0 & 0 & b \\ j & j+a & j+c & j+b \\ j & j+a & j+c & j \end{array} \qquad \begin{array}{llll} 0 & 0 & 0 & c \\ j & j+a & j+b & j+c \\ j & j+a & j+b & j \end{array}$$

$$\begin{array}{llll} 0 & 0 & a & a \\ j & j+a & j+b & j+c \\ j & j+a & j+c & j+b \end{array} \qquad \begin{array}{llll} 0 & 0 & b & b \\ j+a & j+c & j+b & j \\ j+a & j+c & j & j+b \end{array}$$

$$\begin{array}{llll} 0 & 0 & c & c \\ j+a & j+b & j+c & j \\ j+a & j+b & j & j+c \end{array} \qquad \begin{array}{llll} 0 & 0 & a & b \\ j & j+a & j+c & j+b \\ j & j+a & j+b & j \end{array}$$

$$\begin{array}{llll} 0 & 0 & a & c \\ j & j+a & j+b & j+c \\ j & j+a & j+c & j \end{array} \qquad \begin{array}{llll} 0 & 0 & b & c \\ j & j+b & j+a & j+c \\ j & j+b & j+c & j \end{array}$$

$$\begin{array}{llll} 0 & a & b & c \\ 0 & c & a & b \\ 0 & b & c & a \end{array}$$

Case ii: T does not have four leaves.

If T has only two leaves then it would be a path and hence from the labellings of paths P_n where $n \leq 8$ and $n \neq 4$ or 5, and by Theorem 2.4, Lemma 2.6 a V_4-cordial labeling can be obtained. So we can assume that T has exactly three leaves, say l_0, l_1, l_2 connected to v_0, v_1, v_2 respectively. Let v be the unique vertex of T with degree 3. Then at least one of the paths $v-l_0$, $v-l_1$, $v-l_2$ contain at least four edges. Let the path $v-l_0$ contain at least four edges and let v_0' be the other vertex connected to v_0. Remove v_0, l_0, l_1, l_2 and label the resulting tree V_4-cordially. Let the labels on the vertices v_0', v_1, v_2 be respectively a_0', a_1, a_2. Then we can assume, by permuting v_0', v_1 , v_2 and by Lemma 2.1, that (a_0', a_1, a_2) is one of (0, 0, 0), (0, 0, a), (0, 0, b), (0, 0, c), (0,

a, a), (0, b, b), (0, c, c), (0, a, b),(0, b, c), (0, c, a). Suppose that edge-label j appears $m-1$ times while the other three edge-labels appear m times. We must find a way of labeling v_0, l_0, l_1, l_2 with distinct elements so that j appears as an edge-label and no other edge-label appears twice, though j itself might. We do this case by case. Each case is presented as an array with the top row being the a_0', a_1, a_2, the middle row the labels on v_0, l_0, l_1, l_2 and the bottom row the induced edge-labels.

$$\begin{array}{cccc} & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & a & b & c \end{array} \qquad \begin{array}{cccc} & 0 & 0 & a \\ a & b & 0 & c \\ a & c & 0 & b \end{array}$$

$$\begin{array}{cccc} & 0 & 0 & b \\ b & a & 0 & c \\ b & c & 0 & a \end{array} \qquad \begin{array}{cccc} & 0 & 0 & c \\ c & b & 0 & a \\ c & a & 0 & b \end{array}$$

$$\begin{array}{cccc} 0 & a & a & \\ 0 & a & b & c \\ 0 & a & c & b \end{array} \qquad \begin{array}{cccc} 0 & b & b & \\ 0 & b & c & a \\ 0 & b & a & c \end{array}$$

$$\begin{array}{cccc} 0 & c & c & \\ 0 & c & a & b \\ 0 & c & b & a \end{array} \qquad \begin{array}{cccc} 0 & a & b & \\ c & b & a & 0 \\ c & a & 0 & b \end{array}$$

$$\begin{array}{cccc} & 0 & b & c \\ a & c & b & 0 \\ a & b & 0 & c \end{array} \qquad \begin{array}{cccc} & 0 & c & a \\ b & a & c & 0 \\ b & c & 0 & a \end{array}$$

This completes the proof. ∎

Theorem 2.8. *The cycle C_4 and C_5 are not V_4-cordial.*

Proof. Let $v_1v_2\cdots v_n$ denote the cycle C_n. Let f be a V_4-cordial labeling of C_4. Then the vertices of C_4 receive distinct labels under f and induced edge labels are also distinct. To get zero as induced edge label there must be an edge with identical vertex labels at its ends, which is not possible. Suppose f be a V_4-cordial labeling of C_5. Then zero must be an induced edge label of some edge. Without loss of generality let $f(v_1) = f(v_2) = 0$. Then the edges v_1v_5 and v_3v_4 each receive the label $f(v_5)$ and the edges v_1v_2 and v_4v_5 each receive the label $f(v_2)$, which is not possible. ∎

Theorem 2.9. *The cycle C_n is V_4-cordial if and only if $n \neq 4$ or 5 or $n \not\equiv 2 \pmod 4$.*

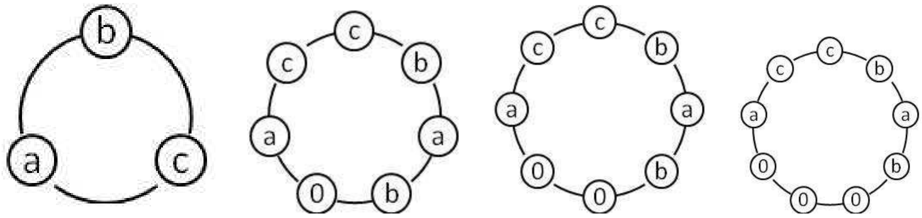


Figure 2:

Proof. First we shall show that C_n is V_4-cordial for $n = 3, 7, 8, 9$. This is verified by the labellings given in Fig. 2.

This with Corollary 2.3, Theorem 2.4 and Theorem 2.8 completes the proof. ∎

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